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# Jauch–Piron Property (Everywhere!) in the Logicoalgebraic Foundation of Quantum Theories

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The Jauch–Piron property of states on a quantum logic is seen to be of considerable importance within the foundation of quantum theories. In this survey we summarize and comment on recent results on the Jauch–Piron property. We also pose a few open problems whose solution may help in further developing quantum theories and noncommutative measure theory.

## PREREQUISITES

*Definition 1.* A quantum logic is a triple $(L, \leq, ')$, where $L$ is a set that is partially ordered by $\leq$ and that fulfills the following requirements:

(i) $L$ possesses a least and a greatest element, 0, 1.
(ii) If $a, b \in L$ and $a \leq b$, then $b' \leq a'$.
(iii) The unary operation $': L \to L$ satisfies the following condition: $(a')' = a$ for any $a \in L$.
(iv) If $a, b \in L$ and if $a \leq b'$, then the supremum $a \vee b$ exists in $L$.
(v) If $a, b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a')$ (the orthomodular law).

Thus, technically speaking, a quantum logic is a formal generalization of the notion of a Boolean algebra (the lattice condition is dropped and the distributivity law is relaxed to the orthomodular law). When a quantum logic is viewed as an event structure of a quantum experiment, the lattice condition on $L$ does not seem to be justified and the distributivity law does seem to be superfluous—its presence could actually bring us outside

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quantum physics. From the mathematical standpoint, too, it seems unduly restrictive to allow lattices only (we would, e.g., lose the logic of projections in a $C^*$-algebra, the logic of skew projections in a Hilbert space, the logic of splitting subspaces in a (noncomplete) inner product space, many interesting set-representable logics, etc.). In this paper, we shall not require logics to be lattices—in fact, sometimes we do not want them to be. Let us denote by $L$ a (quantum) logic throughout this paper.

*Definition 2.* (see, e.g., Pták and Pulmannová, 1991). A mapping $s: L \to \langle 0, 1 \rangle$ is called a *state* on $L$ if it fulfills the following two conditions:

(i) $s(1) = 1$.
(ii) If $a \le b'$, then $s(a \vee b) = s(a) + s(b)$.

Further, a state $s: L \to \langle 0, 1 \rangle$ is called Jauch–Piron (abbreviated a JP state) if the following condition holds true: If $s(a) = s(b) = 1$ for $a, b \in L$, then there is an element $c \in L$ such that $c \le a$, $c \le b$, and $s(c) = 1$. If every state on $L$ is Jauch–Piron, we call $L$ a *Jauch–Piron Logic* (a JP logic).

Unlike the "ordinary" distributive ( = Boolean) case, a logic may not possess any state at all (Greechie, 1971), or—provided it does—none (or most) of them may not be Jauch–Piron (see, e.g., Navara and Rogalewicz, 1991). Note that the Jauch–Piron condition may be thought of in the stochastic fashion—we insist that the pairs of "almost sure" events in a given state admit a subordinated "almost sure" event. Technically, the presence of the Jauch–Piron property may often move us nearer the "classical" ( = Boolean) mathematical areas.

As regards examples of JP logics, such are Boolean algebras (we put $c = a \wedge b$) and the lattices $L(C_n)$ of all projections in an $n$-dimensional Hilbert space ($n \ge 3$). The former statement is obvious and the latter derives as a direct consequence of the famous Gleason theorem (Gleason, 1957). Typically, a logic possesses both JP and non-JP states.

*Example.* Put $\Omega = \langle 0, 1 \rangle^2$ and take for $L$ the collection of all subsets of $\Omega$ whose Lebesgue measure is rational. Thus, $L = \{A \subset \Omega | \mu(A)$ is a rational number$\}$. Then $L$ is a (non-Boolean) quantum logic (we understand $L$ endowed with the inclusion partial ordering and with the set-theoretic orthocomplementation operation ($A' = \Omega - A$). We claim that $L$ is not a JP logic. Indeed, take a measurable subset of $\Omega$, some $B$, with $\mu(B) > 0$, and define a state $s: L \to \langle 0, 1 \rangle$ by putting $s(A) = s(A \cap B)/s(B)$ $(A \in L)$. Then $s$ is Jauch–Piron if and only if $\mu(B)$ is a rational number.

Let us now examine miscellaneous aspects of Jauch–Pironness.

## 1. "DISCRETE" JAUCH–PIRON LOGICS

Let us start with finite logics. Even in this class we have the "Greechie phenomenon" to be aware of—there are finite logics without any state at all (Greechie, 1971). Obviously, a stateless logic is also Jauch–Piron by our definition, but we are naturally more interested in logics whose states spaces are reasonably rich. Let us call a logic $L$ *unital* (Gudder, 1979) if the following condition is fulfilled: If $a \in L$ and if $a \neq 0$, then there is a state $s$ on $L$ such that $s(a) = 1$. We now have the following result.

*Theorem 1.1.* (Rüttimann, 1977). Let $L$ be a finite unital Jauch–Piron logic. Then $L$ is Boolean.

This result (resp. the appeal of the way in which it was proved) considerably contributed to the increase of the interest in JP logics. Rüttimann (1977) seems also to be responsible for the name of this class of logics.

The effort to generalize Theorem 1.1 for "simple" infinite logics resulted in the following two theorems.

*Theorem 1.2.* (Rogalewicz, 1991). Let $L$ be a unital Jauch–Piron logic. Let $L$ contain only finitely many maximal Boolean subalgebras. Then $L$ is Boolean.

*Theorem 1.3.* (Ovchinnikov, 1991). There is a unital countable Jauch–Piron logic that is not Boolean. Moreover, the latter logic can be required a sublogic of the projection logic $L(C_3)$.

To complete the schema here, it seems desirable to know if one can construct Greechie logics fulfilling the properties of Theorem 1.3. (Let us call a logic Greechie if it is atomic and every two maximal Boolean subalgebras in it meet in at most one atom.) We do not know the answer to this question.

In the conclusion of this section, let us note that recently the following strengthening of the JP condition has appeared (de Lucia and Pták, 1992; Majerník and Pulmannová, 1992; Pták and Pulmannová, n.d.). Let us say that a state $s$ on $L$ is strongly Jauch–Piron if for any couple $a, b \in L$ there is an element $c \in L$ such that $c \geq a$, $c \geq b$, and $s(c) \leq s(a) + s(b)$. Obviously, if $s$ is strongly Jauch–Piron, then it is Jauch–Piron. The "*vice versa*" statement does not hold: Every lattice logic that is unital with respect to strongly Jauch–Piron states has to be Boolean (Pták and Pulmannová, n.d.).

## 2. THE JAUCH–PIRON PROPERTY IN CONCRETE LOGICS

A logic is called *concrete* if it can be represented by a collection of subsets of a set. In other words, $L$ is concrete if $L \subset \exp S$, where $\exp S$ is the collection of all subsets of a set $S$, and if the following conditions are satisfied:

(i) $\emptyset \in L$.
(ii) If $A \in L (A \subset S)$, then $S - A \in L$.
(iii) If $A, B \in L (A, B \subset S)$ and if $A \cap B = \emptyset$, then $A \cup B \in L$.

Thus, the concrete logics are in a sense "nearly Boolean." [It should be noted that such (or very similar) structures appeared already in the classics of the descriptive theory of sets and mathematical analysis many years ago (see, e.g., Kuratowski, 1966).] The conceptual value of concrete logics for quantum axiomatics seems first to be pointed out by Gudder (1969, 1979).

When does a concrete Jauch–Piron logic have to be Boolean? The next result (one of the first results in this line) says that it is so quite often and that it is always "nearly" so.

*Theorem 2.1* (Navara and Pták, 1989). Let $L$ be a concrete Jauch–Piron logic. Then the following statements hold true:

(i) If $L$ is a lattice, then $L$ is Boolean.
(ii) If $L$ lives on an at most countable set [i.e., if $L = (S, L_S)$, where $\Omega$ is at most countable], then $L$ is Boolean.
(iii) If $L \subset \exp S$ for a set $S$ and if $A, B \in L (A, B \subset S)$, then there is a finite collection $\{C_1, C_2, \ldots, C_n\} \subset \exp S$ such that $C_i \in L$ for any $i$ $(i \leq n)$ and $A \cap B = \bigcup_{i \leq n} C_i$.

Let us pause for a moment at the condition (iii) to acquire better insight into the kind of the problems that are pursued here. The proof of the condition (iii) goes approximately as follows: If $A \cap B \neq \emptyset$, then there is a state $s$ on $L$ such that $s(A) = s(B) = 1$. Thus, the set $\mathscr{S}_{A,B} = \{t$ is a state on $L | t(A) = t(B) = 1\}$ is nonvoid. Moreover, $\mathscr{S}_{A,B}$ is compact in the pointwise topology. For any $C \in L$ with $C \subset A \cap B$, put $\mathscr{S}_C = \{t \in \mathscr{S}_{A,B} | s(C) > 0\}$. Since $L$ is Jauch–Piron, we have $\mathscr{S}_{A,B} = \bigcup \mathscr{S}_C$, where $C$ varies over all sets $C \in L$ such that $C \subset A \cap B$. Since every $\mathscr{S}_C$ is open in $\mathscr{S}_{A,B}$, we let the compactness of $\mathscr{S}_{A,B}$ work for us to arrive at a finite family $C_i$ $(i \leq n)$ with $\mathscr{S}_{A,B} = \bigcup_{i \leq n} \mathscr{S}_{C_i}$. Obviously, $\bigcup_{i \leq n} C_i = A \cap B$.

Let us come back to the question which was asked prior to the latter theorem. This question appeared to be fairly nontrivial. However:

*Theorem 2.2* (Müller, 1993). There is a concrete Jauch–Piron logic that is *not* Boolean.

In fact, the technique utilized in the latter result guarantees a proper class of Jauch-Piron logics that are not Boolean. The following question then announces itself immediately. Can every concrete logic be embedded (in a compatibility-preserving manner) into a concrete Jauch-Piron logic? The answer to this question seems to be unknown for the time being.

*Remark*. Bunce *et al.* (1985) succeeded in solving the $\sigma$-complete version of the question posed above. They solved it in the affirmative, needing, however, the set-theoretic assumption TRM of the nonexistence of real-measurable cardinals (can this set-theoretic assumption be omitted?).

## 3. JAUCH-PIRONNESS IN PROJECTION LOGICS

The study of the JP condition in the logics of projections in von Neumann algebras started with Amann (1987) and was further deepened in Bunce and Hamhalter (n.d.) and Hamhalter (1993). Since a more detailed exposition of these results is contained in the present volume, let us only state here two results which are directly related to the contents of this survey.

*Theorem 3.1* (Hamhalter, 1993). Let $\mathscr{A}$ be a von Neumann algebra. Then the logic $\mathscr{P}(\mathscr{A})$ of all projections of $\mathscr{A}$ is Jauch-Piron if and only if $\mathscr{A}$ is a direct sum of a commutative von Neumann algebra and a finite-dimensional von Neumann algebra.

In the projection logics, an interesting line of investigation presents also the "individual" Jauch-Piron condition. For instance, the following elegant result is in force [the result may have a direct interpretation in quantum foundations (see e.g., Bugajski *et al.*, n.d.; Emch, 1972).

*Theorem 3.2* (Bunce and Hamhalter, n.d.). Let $\mathscr{A}$ be a von Neumann algebra which does not contain a central Abelian part. Let $s$ be a pure state on $\mathscr{P}(\mathscr{A})$. Then $s$ is Jauch-Piron if and only if $s$ is $\sigma$-additive.

## 4. A LINK OF JAUCH-PIRON PROPERTY WITH TOPOLOGICAL REPRESENTATIONS OF LOGICS

In an analogy with Boolean algebras, a natural project is to look for set or topological representations of logics. Obviously, the standard Stone representation technique cannot be adopted here—the set of two-valued states on a logic may be very poor (see, e.g., Greechie, 1971). Several attempts have been made to obtain at least some weaker representations of logics (or orthomodular lattices); see, e.g., Binder and Pták (1990), Itturioz (1986), Pták (1983), and Tkadlec (1991, 1993). An interesting topological representation was found in Tkadlec (1991) and, to a certain surprise, the Jauch–Piron condition appears again [see Tkadlec (1993) for a precise definition of all notions; see also Tkadlec (1991) for relevant comments and open problems].

*Theorem 4.1* (Tkadlec, 1993). Let $L$ be a logic. Then there is a zero-dimensional closure space $(S, -)$ such that $L$ can be order-orthoembedded in the orthomodular lattice of clopen subsets of $S$. Moreover, $S$ can be taken a topological space if and only if $L$ possesses a unital set of weakly additive Jauch–Piron states.

## 5. CAN THE JAUCH–PIRON CONDITION HELP IN EXTENDING STATES?

Let us call $L$ *state universal* if the following implication holds true: Whenever $L$ is embedded in $K$, where $K$ is a unital logic, every state on $L$ can be extended over $K$. We do not know whether every Jauch–Piron logic is state universal [though in the "most natural" cases to be tested it is so (Hamhalter, n.d.; Pták, 1985). It should be observed that, on the other hand, there are state universal logics which are not Jauch–Piron [such as, e.g., $L(H)$ for dim $H = \infty$; see Theorems 3.1 and 3.2].

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